Pigeonhole Principle

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1. a) The basic pigeonhole principle says that if you have n things and k categories (say, n pigeons and k holes), and if $n > k$, there must be at least one category that contains at least two things. Prove it! Can you prove it without using a proof by contradiction?

   b) The generalized pigeonhole principle says that if you have n things and k categories and $n > jk$ then there must be at least one category with at least $j+1$ things in it. Prove this.

   Sometimes the pigeonhole principle is called the Dirichlet box principle.

2. Prove that if you pull three socks out of a drawer containing socks of two different colors, you must be holding at least one pair of socks.

   Generalize! (E.g., you need 3 pairs for your trip, how many socks do you need to choose? What if there are three colors in the drawer?)

3. Prove that if you throw five darts at a square target measuring 14 inches on a side, two darts must be within 10 inches of each other.

   What if it's an equilateral triangle 14 inches on a side (what number in place of the 10, then?) What if it's 4 darts on the triangle?

4. Prove that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.

5. Prove that if you choose 51 numbers from 1 through 100, two of them must add to 101.

   Prove that if you choose 55 integers from 1 through 100, there must be a pair that differs by 9, a pair that differs by 10, and also 12, and 13 … but not necessarily 11!

   Prove that if you choose 51 numbers from 1 through 100, there must be two of them (not necessarily distinct) that add to a third. Also prove that 50 is not enough.

   Many more variations here are possible…

6. Prove that if you rearrange the numbers 1 through n, in some order, and then subtract 1 through n from each in that order, and then multiply, the result is odd. Oh, wait, you can disprove it? OK, you're right. Prove it if n is even.

7. Prove that every integer has a multiple that consists only of 0s and 1s.

8. There are n people in the room; some are mutual friends and the rest are mutual enemies. Prove that there are 2 people with the same number of friends (nobody is friends with themselves).

   There are 6 people in the room; prove that there must be a set of three such that either all three are mutual friends, or all three are mutual enemies.

   There are ??? people in the room; prove that there must be a set of four such that either all four are mutual friends, or all four are mutual enemies.

9. Prove that in 8D 3x3x…x3 tic-tac-toe, a draw is impossible even if the players cooperate, are allowed to skip turns, and so on.

10. In any permutation of the numbers 1 through 10, prove that there exists either an increasing subsequence of at least four numbers, or a decreasing subsequence of at least four numbers, or both. Why?

    Generalize!

11. 41 rooks are placed on a 10x10 chessboard. Prove that you can remove 36 of the rooks and leave 5 rooks none of which can attack each other.