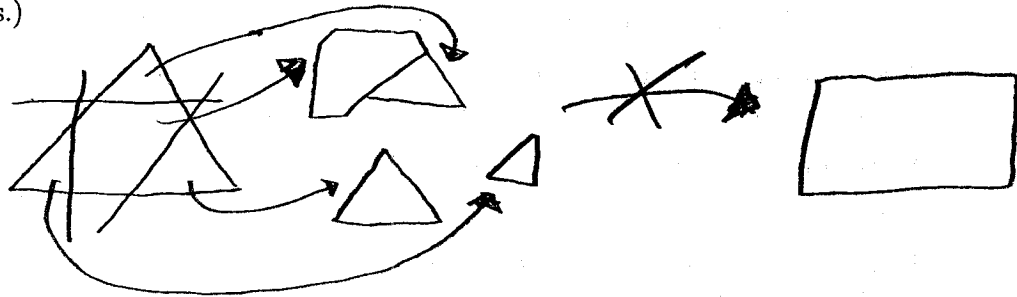


# OEBMC TEACHER'S CIRCLE: PROBLEM SOLVING TECHNIQUES

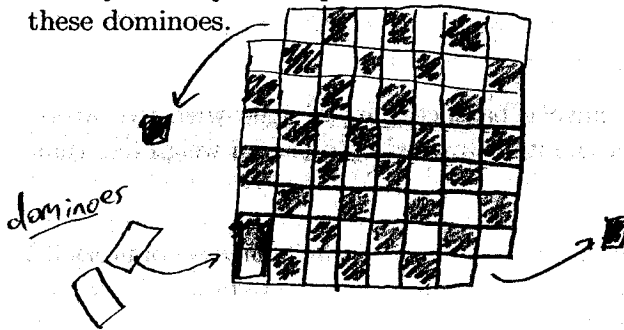
CHARLES CRISSMAN

## 1. INVARIANTS

**Problem 1.** \*\*\* Suppose we start with a triangle, and we cut our triangle into finitely many pieces using straight lines, then use *only* translations to slide these pieces around and reassemble them. Show that it is not possible to form a rectangle in this way. (To make this problem easier, show that it is not possible to form a rectangle with sides parallel to the coordinate axes.)



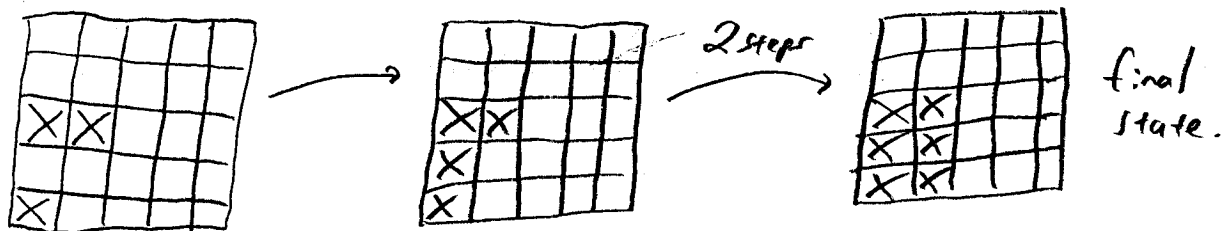
**Problem 2.** \*\* Suppose we take an  $8 \times 8$ -chessboard and cut off two opposite corners. We have some rectangular dominoes, each of which can cover exactly two adjacent squares. Show that we cannot tile the chessboard with these dominoes.



**Problem 3.** \*\*\* On a  $5 \times 5$ -grid, we begin with some squares with  $X$ 's in them and the rest of the squares blank. If any empty square shares at least 2 sides with squares with  $X$ 's in them, then we draw an  $X$  in this empty square as well (see picture). We repeat this until every square either has an  $X$  in it or shares at most one edge with such a square.

Date: September 17th, 2007.

Show that if we start with a grid with less than five X's in it, then when we finish this process, some squares will still be blank.



## 2. THE PIGEONHOLE PRINCIPLE

**Problem 1.** \* What is the maximum number of rooks one can place on an  $8 \times 8$ -chessboard such that no rook can capture any other?

**Problem 2.** \*\* What is the maximum number of bishops one can place on an  $8 \times 8$ -chessboard such that no bishop can capture any other?

**Problem 3.** \*\* (Putnam 2002) Show that if five points lie on the surface of a sphere, then there is some hemisphere containing four of them.

**Problem 4.** \*\*\* Given five points in the unit square, show that there is some pair of points whose distance is less than or equal to  $\frac{\sqrt{2}}{2}$ .

**Problem 5.** \* Suppose we color the  $x, y$ -plane two colors: each point  $(x, y)$  is colored either red or blue. Show that there exist two points of the same color which are distance exactly 1 apart.

**Problem 6.** \*\*\*\* Suppose now that we color the  $x, y$ -plane three colors: each point  $(x, y)$  is colored either red, blue, or green. Show that there exist two points of the same color which are distance exactly 1 apart. (This also uses the technique of *proof by counterexample*.)

**Problem 7.** \*\* Suppose we have a balance (i.e. a scale with two arms, which can only tell us whether the items placed on one arm weigh less than or more than the items placed on the other arm). We are given 9 coins, one of which is counterfeit and weighs more than its real counterparts. Using only two weighings, is it possible to determine which of the 9 coins is counterfeit? Suppose instead we start with 10 coins; is it now possible to determine which is counterfeit in only two weighings?

## 3. MODULAR ARITHMETIC AND RELATIVE PRIMENESS

**Problem 1.** \* Suppose we have two drinking glasses, one which holds 21 oz. and one which holds 13 oz. Is it possible to measure exactly 1 oz. of water using only these two cups? What if our two cups are 8 oz. and 10 oz.?

**Problem 2.** \*\* Suppose we begin with a knight sitting at  $(1, 0, 0)$  in three-dimensional space. A *three-dimensional knight jump* is a move where a piece goes  $\pm 1$  along one axis (i.e. in the  $x$ ,  $y$ , or  $z$ -direction),  $\pm 2$  along a second axis, and  $\pm 3$  along the remaining axis. For example, our knight at  $(1, 0, 0)$  could make a three-dimensional knight jump to  $(1, 0, 0) + (2, -1, 3) = (3, -1, 3)$ , or perhaps to  $(1, 0, 0) + (-1, -3, -2) = (0, -3, -2)$ . Show that it is not possible for our knight to reach  $(0, 0, 0)$  by making a finite number of three-dimensional knight jumps.

**Problem 3.** \*\* Find all prime numbers  $p$  such that  $p + 2$  and  $p + 4$  are also prime.

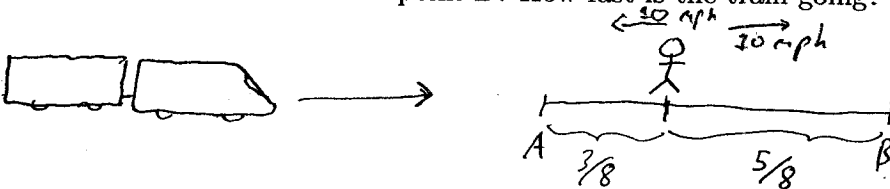
**Problem 4.** \*\*\* Suppose  $a_1, a_2, \dots, a_{14}$  are some 14 integers. Show that

$$\prod_{1 \leq i < j \leq 14} (a_i - a_j)$$

is divisible by  $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ .

#### 4. SYMMETRY

**Problem 1.** \* There is a railroad bridge stretching from point  $A$  to point  $B$ . A man is standing on the bridge  $3/8$  of the way across from point  $A$  (so  $5/8$  of the length of the bridge away from point  $B$ ). The man hears a train coming from the direction of point  $A$ , and he begins running toward either point  $A$  or point  $B$  at a speed of 10 miles per hour. It turns out that if he runs towards point  $A$ , the train will hit him just as he reaches point  $A$ , and if he runs toward point  $B$ , the train will likewise catch him just as he reaches point  $B$ . How fast is the train going?



**Problem 2.** \*\*\* Alice and Bob play the following game: they draw a perfect circle on a flat table, then they take turns placing quarters within this circle. No two quarters are allowed to overlap. A player loses when they can no longer place a quarter within the circle. Alice, of course, goes first. Does either player have a winning strategy?