

SPLIT A PILE

In all the games described below there are two players, Alice and Bob, and Alice always plays first. The problem is to decide which one of the two players has a winning strategy (and, of course, to describe this strategy).

We recommend that you start with the first game, play it several times and then answer all the questions. Then go to the next game and do the same. The games become more challenging (and more interesting!). To play the games, you may use the provided plastic chips instead of pebbles.

An answer to the question "Which player has a winning strategy?" must include a detailed description of such strategy, i.e., you have to explain what the winning player should do so that this player wins REGARDLESS of his opponent's moves. See the example below for further explanation. To *solve* a game means to find a winning or a non-losing strategy for one of the players. If you cannot solve a game but you found some interesting patterns, or you solved a game for some special cases different from those specified, please describe them for bonus points.

Example:

There are 25 matches on a table. During each turn, a player may take any number of matches between 1 and 4. The player who takes the last match wins.

Solution: In this game, Bob has a winning strategy. After each of Alice's moves he needs to take $5 - x$ matches if Alice took x matches. For example, if Alice took 3 matches in her first move, Bob takes 2 matches. Now there are 20 matches left. Clearly, if Bob continues in that way, the number of matches left before Alice's next move will be 15, 10, and 5. Now, no matter what she does, Bob will win.

GAMES:

1. Players start with one pile of pebbles. During each move, a player must split exactly one pile into two nonempty piles. The player unable to make a move loses.

Questions:

- (i) Which player has a winning strategy if they start with 2 pebbles? 3 pebbles? 4 pebbles? 15 pebbles?
- (ii) Can you solve the game in general?

2. Players start with one pile of pebbles. During each move, a player must split one pile into two nonempty piles. Piles that contain one or two pebbles cannot be split. The player unable to make a move loses.

Questions:

- (i) Which player has a winning strategy if they start with 3 pebbles? 4 pebbles? 5 pebbles? 6 pebbles? 7 pebbles? 8 pebbles?

(ii) Which player has a winning strategy if they start with any even number of pebbles?

3. Players start with one pile of pebbles. During each move, a player must split one pile into two nonempty piles in such a way that all resulting piles have different number of pebbles. The player unable to make a move loses.

Questions:

(i) After the first move, the piles contain 5 and 11 pebbles. Find a winning strategy for Bob.

(ii) After the first move, the piles contain 5 and 11 pebbles. Give an example of a bad move after which Bob will necessarily lose.

(iii) Which player has a winning strategy if they start with 11 pebbles?

(iv) Which player has a winning strategy if they start with 22 pebbles?

(v) Can you solve the game in general?